A Chebyshev Approximation Method for Microstrip Problems

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Abstract—The quasi-static TEM mode of a microstrip line may be obtained approximately from the solution of Laplace's equation subject to certain boundary conditions. The Green's function approach leads to the solution of a Fredholm integral equation with a logarithmic singularity in the kernel. It is shown that if the charge distribution on the strip is expanded in terms of Chebyshev polynomials then the integrals arising from the logarithmic term may be evaluated in closed from, and the integral equation may be approximated closely by a set of algebraic equations. The method is applied to numerous open and shielded configurations of strips and couple-strips in the presence of dielectrics. Numerical results are compared with exact results whenever possible and with results from previous authors. Design curves are presented for particular shielded couple-strip configurations.

I. INTRODUCTION

THE lowest order quasi-TEM mode of a microstrip line may be obtained approximately from the solution of Laplace's equation subject to certain boundary conditions. If a Green's function approach is used the problem may be reduced to the solution of a Fredholm integral equation of the first kind. For a single strip occupying the interval $-w/2 \leq x_0 \leq w/2$ the equation is

$$\int_{-w/2}^{w/2} G_0(x_0,\xi_0)\sigma_0(\xi_0)\, d\xi_0 = f_0(x_0), \qquad -w/2 \leq x_0 \leq w/2. \tag{1}$$

Here x_0,y_0 are the actual dimensional coordinates; $G_0(x_0,\xi_0)$ is the Green's function; $\sigma_0(\xi_0)$ is the charge distribution, and $f_0(x_0)$ is the potential on the strip. If $f_0(x_0)$ is constant and unity, then the capacitance of the strip is the total charge.

For computational purposes it is convenient to introduce dimensionless variables x, ξ given by $x = 2x_0/w$, $\xi = 2\xi_0/w$. It is known that $G_0(x_0,\xi_0)$ will have a logarithmic singularity $\ln|x_0 - \xi_0|$. Thus $G_0(x_0,\xi_0)$ may be written

$$\begin{aligned} G_0(x_0,\xi_0) &= \nu_0^{-1}[\ln|x_0 - \xi_0| + H_0(x_0,\xi_0)] \\ &= \nu_0^{-1}[\ln|x - \xi| + H(x,\xi)] \end{aligned} \tag{2}$$

where $H_0(x_0,\xi_0)$ and $H(x,\xi)$ are continuous on the strip and ν_0 is a function of the geometry and electrical constants. Thus (1) may be written

$$\int_{-1}^{1} G(x,\xi)\sigma(\xi)\, d\xi = \nu f(x) \tag{3}$$

where

$$\nu = 2\nu_0/w, \qquad \sigma(\xi) = \sigma_0(\xi_0), \qquad f(x) = f_0(x_0)$$

and

$$G(x,\xi) = \ln|x - \xi| + H(x,\xi). \tag{4}$$

Equation (3) will be treated as the standard form of (1). It is known that the charge distribution $\sigma_0(\xi_0)$ will have a square root singularity at the ends of the strip. This means that $\sigma(\xi)$ will have a singularity of the form $(1 - \xi^2)^{-1/2}$.

Numerous methods have been proposed for the solution of (3). It may be solved by the method of subareas [1] and the method of moments [2]. These methods ignore the singularity in $\sigma(\xi)$ at the ends but, nevertheless, yield acceptable results. Another method, proposed by Silvester and Benedek [3], uses Gaussian quadrature with weight $(1 - \xi^2)^{-1/2}$ as given by Stroud and Secrest [4] together with a special Gaussian quadrature for the logarithm. This method was applied to a single microstrip, but was valid only for a restricted range of parameters, and led to a 2-percent capacitance error.

The method proposed in this paper takes account of the singularity in $\sigma(\xi)$, deals exactly with the logarithmic singularity in the kernel, uses only ordinary Gauss–Chebyshev quadrature, is simple and accurate, and can be applied to any microstrip configuration.

II. CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials $T_i(x)$, $U_i(x)$ of the first and second kinds, respectively, are defined [5] by the equations

$$T_i(x) = \cos i\theta,\ U_i(x) = [\sin(i+1)\theta]/\sin\theta, \qquad x = \cos\theta, \qquad i = 0,1,2\cdots \tag{5}$$

and are polynomials of degree i in x. The first few are

$$T_0(x) = 1 \quad T_1(x) = x \quad T_2(x) = 2x^2 - 1$$
$$U_0(x) = 1 \quad U_1(x) = 2x \quad U_2(x) = 4x^2 - 1.$$

The following properties will be used: the orthogonality condition

Manuscript received January 27, 1975; revised June 6, 1975. This work was supported by the National Research Council of Canada.
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$$\frac{2}{\pi}\int_{-1}^{1} \frac{T_i(x)T_j(x)\,dx}{(1-x^2)^{1/2}} = \begin{cases} 0, & i \neq j \\ 1, & i = j \neq 0 \\ 2, & i = j = 0 \end{cases} \tag{6}$$

the special integral

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\ln|x-\xi|\,T_i(\xi)\,d\xi}{(1-\xi^2)^{1/2}} = \begin{cases} -T_i(x)/i, & i \neq 0 \\ -\ln 2, & i = 0 \end{cases} \tag{7}$$

and the Gauss–Chebyshev quadrature formula

$$\int_{-1}^{1} \frac{p(x)}{(1-x^2)^{1/2}} \simeq \frac{\pi}{n+1}\sum_{k=1}^{n+1} p(x_k), \qquad x_k = \cos\frac{(2k-1)\pi}{2n+2} \tag{8}$$

which is exact for polynomials of order $2n+1$ or less.

Equation (6) follows from the definition (5) and the orthogonality of the functions $\cos i\theta$ over $[0,\pi]$. Equation (7) can be deduced from the two relations

$$\int_{-1}^{\xi} \frac{T_i(t)\,dt}{(1-t^2)^{1/2}} = -\frac{(1-\xi^2)^{1/2}U_{i-1}(\xi)}{i}, \qquad i \neq 0, |\xi| \leq 1 \tag{9}$$

and

$$(P)\int_{-1}^{1} \frac{(1-\xi^2)^{1/2}U_{i-1}(\xi)\,d\xi}{\xi - x} = -\pi T_i(x), \qquad i \neq 0, |x| \leq 1. \tag{10}$$

Equation (9) follows immediately from (5), while (10) is given in [6]; (P) denotes the principal value of the integral.

III. THE SOLUTION

Write

$$\sigma(\xi) = \psi(\xi)/(1-\xi^2)^{1/2} \tag{11}$$

and assume that

$$\psi(\xi) = \sum_{i=0}^{n} a_i T_i(\xi) \tag{12}$$

where the a_i are coefficients to be determined. Then (7) shows that

$$\int_{-1}^{1} \frac{\ln|x-\xi|}{(1-\xi^2)^{1/2}}\psi(\xi)\,d\xi = -\pi \sum_{i=0}^{n}{}^{*}\, a_i T_i(x)/i \tag{13}$$

where the asterisk indicates that the first term is $a_0 \ln 2$. If $G(x,\xi)$ has the form (4), then (13) gives the first part of the integral in (3). The second part may be obtained by using the Gauss–Chebyshev formula (8). Thus

$$\int_{-1}^{1} H(x,\xi)\sigma(\xi)\,d\xi = \sum_{j=0}^{n} a_j \int_{-1}^{1} \frac{H(x,\xi)T_j(\xi)\,d\xi}{(1-\xi^2)^{1/2}}$$

which may be approximated by

$$\int_{-1}^{1} H(x,\xi)\sigma(\xi)\,d\xi \simeq \pi \sum_{j=0}^{n} a_j b_j(x) \tag{14}$$

where

$$b_j(x) = \frac{1}{n+1}\sum_{k=1}^{n+1} H(x,\xi_k)T_j(\xi_k),$$

$$\xi_k = \cos\left[\frac{(2k-1)\pi}{2n+2}\right]. \tag{15}$$

This is equivalent to approximating $H(x,\xi)$ by a polynomial of degree n in ξ with coefficients that are functions of x.

Equation (3) now becomes

$$-\pi \sum_{i=0}^{n}{}^{*}\, a_i T_i(x)/i + \pi \sum_{j=0}^{n} a_j b_j(x) = \nu f(x). \tag{16}$$

This equation may be solved approximately by equating both sides for a number of values of x or equivalently by employing another Gauss–Chebyshev integration. The latter yields the equations

$$-\pi a_0 \ln 2 + \pi \sum_{j=0}^{n} c_{ij}a_j = \nu d_0, \qquad i = 0 \tag{17}$$

$$-\pi a_i/2i + \pi \sum_{j=0}^{n} c_{ij}a_j = \nu d_i, \qquad i = 1,2,\cdots,n \tag{18}$$

where

$$c_{ij} = \frac{1}{n+1}\sum_{k=1}^{n+1} T_i(x_k)b_j(x_k)$$

$$d_i = \frac{1}{n+1}\sum_{k=1}^{n+1} T_i(x_k)f(x_k). \tag{19}$$

Equations (17) and (18) are $(n+1)$ equations for the determination of the $(n+1)$ coefficients $a_0,a_1,\cdots,a_n$.

The particular case of (3) in which $H(x,\xi) \equiv 0$ is governed by Carleman's formula [7]. This states that

$$\psi(\xi) = \frac{\nu}{\pi^2} \cdot \left[(P)\int_{-1}^{1} \frac{(1-t^2)^{1/2}}{t-\xi} f'(t)\,dt - \frac{1}{\ln 2}\int_{-1}^{1} \frac{f(t)\,dt}{(1-t^2)^{1/2}}\right]. \tag{20}$$

The integration of $f(x)$ used to derive (17) and (18) from (16) is equivalent to approximating $f(x)$ by the polynomial

$$f(x) = d_0 + 2\sum_{i=1}^{n} d_i T_i(x). \tag{21}$$

In this case, since $T_i'(x) = iU_{i-1}(x)$

$$f'(t) = 2\sum_{i=1}^{n} i d_i U_{i-1}(t) \tag{22}$$

so that (10) gives

$$\psi(\xi) = -(\nu/\pi)\{2\sum_{i=1}^{n} id_iT_i(\xi) + d_0/\ln 2\} \qquad (23)$$

which is the result obtained from (17) and (18) when $c_{ij} = 0$.

IV. IMPLEMENTATION

In practice the solution is carried out in the following steps.

1) Find the Green's function for the problem.
2) Scale the variables so that the strip occupies the region $[-1,1]$.
3) Separate the logarithmic singularity from the Green's function and write it in the form of (2).
4) Choose n and find the coefficients c_{ij},d_i from (15) and (19). It has been found that $n = 5$ is more than adequate in every problem so far attempted.
5) Solve (17) and (18).

The charge distribution is given by

$$\sigma(\xi) = \psi(\xi)/(1-\xi^2)^{1/2}$$

where $\psi(\xi)$ is given by (12). In particular, the total charge, obtained by integrating $\sigma(\xi)$ over $[-1,1]$ is, by (6) with $j = 0$, given by πa_0. Notice, therefore, that no extra integration is needed for the computation of the total charge.

The most difficult part of the analysis is step 3. This will be discussed in examples below.

V. EXAMPLES

The simplest problem is the microstrip *in vacuo*, obtained by putting $\epsilon_1 = \epsilon_0$ in Fig. 1, for which Palmer [8] obtained an exact solution based on conformal mapping. Here the scaled Green's function $G(x,\xi)$ is [3]

$$G(x,\xi) = \ln|x-\xi| - \tfrac{1}{2}\ln\{(x-\xi)^2 + (4H/W)^2\} \qquad (24)$$

and $\nu = -4\pi\epsilon_0/W$ in (3).

Table I shows values of the capacitance per unit length

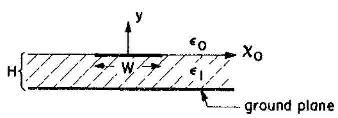

Fig. 1. The geometry of the open microstrip.

TABLE I
APPROXIMATE VERSUS EXACT CAPACITANCE PER UNIT LENGTH OF MICROSTRIP LINE IN VACUO

W/H	[pF/m] by present method			[pF/m] by Palmer
	N=2	N=4	N=6	exact
0.0983	12.647	12.647	12.647	12.651
0.2120	15.318	15.318	15.318	15.322
2.3421	41.116	41.056	41.057	41.069
4.3533	61.815	61.269	61.266	61.285
9.5915	126.47	112.53	111.26	111.16

obtained from the present method and from Palmer's, and it will be noted that for $n = 6$ the error is less than 0.1 percent for all values of (W/H). Palmer's analysis involves elliptic integrals; an initial step in his analysis is the choice of a parameter k and the determination of the ratio W/H corresponding to it: this accounts for the particular W/H values chosen in Table I. Palmer's method is entirely unsuited to computation; his results take considerably more time to compute and are, in the authors' opinion, subject to greater error than the values computed by the present method. The present method also immediately gives the charge distribution which cannot be deduced from Palmer's results. When applied to the configuration of Fig. 1 with $\epsilon_1 \neq \epsilon_0$ the method was found to give results in close agreement with those in [3]. It was found also that results obtained by using different values of n converged rapidly.

Farrar and Adams [9] have considered the shielded microstrip shown in Fig. 2. For general B/H the Green's function may be obtained in the form of (4) by using the procedure suggested by Coen [10]. Farrar and Adams derived the Green's function for integral B/H; in particular for $B/H = 2$ they give the scaled Green's function

$$G(x,\xi) = -\sum_{m=1}^{\infty} \frac{1}{m} \exp\{-m\pi|x-\xi|W/4H\}, \qquad m \text{ odd} \qquad (25)$$

but incorrectly omit the restriction that m is odd. The parameter $\nu = -(\epsilon_0 + \epsilon_1)/W$ in (3). The series may be summed explicitly and written in the form

$$G(x,\xi) = \ln|x-\xi| - \ln\{|x-\xi|\coth[\pi|x-\xi|W/8H]\} \qquad (26)$$

in which the second term is continuous throughout the interval. The expression (26), which is apparently new, is more convenient than (25) for computation. For general values of B/H, Coen's method immediately gives $G(x,\xi)$ with the logarithmic term separated out, as shown in the Appendix. The exact capacitance for the configuration $B/H = 2$ was obtained by Oberhettinger and Magnus [11] and as noted by Cohn [12] in the form

$$C/\epsilon_{\text{eff}} = 4K[\tanh \pi W/4H]/K[\text{sech}\, \pi W/4H] \qquad (27)$$

where K is the complete elliptic integral of the first kind and

$$\epsilon_{\text{eff}} = (\epsilon_r + 1/2)\epsilon_0, \qquad \epsilon_r = \epsilon_1/\epsilon_0.$$

The results obtained by the present analysis applied to the kernel (26), $n = 4$, and $W/H = 0.02$, 0.2, and 2.0 were found to agree with those obtained from (27) to at

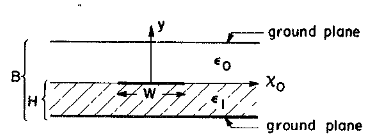

Fig. 2. The geometry of the shielded microstrip.

least 5 decimal places; for $W/H = 0.2$ the two values agreed to 10 decimal places!

In order to compare the approximate charge distributions given by (11) and (12) with previously published approximate results, the three shielded strips considered by Mittra and Itoh [13] were computed. In each case the graphs were found to be identical to theirs.

Cohn [14] also considered the shielded couple-strip shown in Fig. 3, and obtained exact results for the impedance for the even and odd modes. The analysis described in Section III may easily be extended to this case. Before the variables x_0,ξ_0 are scaled, there are two integral equations which state that the potential at point x_0 of either strip is produced by the distributions of charge on the pair of strips. The equations may be written

$$\int_{-b}^{-a} G_0(x_0,\xi_0)\sigma_{10}(\xi_0)\, d\xi_0 + \int_a^b G_0(x_0,\xi_0)\sigma_{20}(\xi_0)\, d\xi_0 = f_{j0}(x_0) \tag{28}$$

where $j = 1$ refers to x_0 on the left-hand strip; $j = 2$ refers to x_0 on the right hand. These equation are brought to the required form as follows. For each value of j, in turn, the x_0 variable is scaled to run from -1 to 1, and the ξ_0 variable in each integral is scaled to run from -1 to 1 also. Thus for $j = 1,2, x_0$ is written, respectively, in the forms

$$x_0 = [(b-a)x' - (b+a)]/2$$
$$x_0 = [(b-a)x'' + (b+a)]/2$$

and the ξ_0 variables in the two integrals are written

$$\xi_0 = [(b-a)\xi' - (b+a)]/2$$
$$\xi_0 = [(b-a)\xi'' + (b+a)]/2$$

respectively. Thus (28) becomes

$$\int_{-1}^{1} G_{11}(x',\xi')\sigma_1(\xi')\, d\xi' + \int_{-1}^{1} G_{12}(x',\xi'')\sigma_2(\xi'')\, d\xi'' = \nu_1 f_1(x'), \qquad -1 < x' < 1 \tag{29}$$

$$\int_{-1}^{1} G_{21}(x'',\xi')\sigma_1(\xi')\, d\xi' + \int_{-1}^{1} G_{22}(x'',\xi'')\sigma_2(\xi'')\, d\xi'' = \nu_2 f_2(x''), \qquad -1 < x'' < 1 \tag{30}$$

where ν_1,ν_2 are constants depending on the geometry.

In (28) for $j = 1$ the kernel $G_0(x_0,\xi_0)$ has a logarithmic singularity, but this needs to be considered only in the first integral; in the second integral x_0 lies on strip 1, ξ_0 lies on strip 2 so that $x_0 - \xi_0$ is never zero. Now since $x_0 - \xi_0 = (b-a)(x'-\xi')/2$, the kernel $G_{11}(x',\xi')$ will have a singularity $\ln|x'-\xi'|$; $G_{12}(x',\xi'')$ can be treated as continuous. In the same way, $G_{22}(x'',\xi'')$ has a singularity $\ln|x''-\xi''|$ while $G_{21}(x'',\xi')$ is continuous. In addition, if $\sigma_{10}(\xi),\sigma_{20}(\xi)$ have square-root singularities at the ends of the respective strips, then $\sigma_1(\xi'),\sigma_2(\xi'')$ will have the forms

$$\sigma_1(\xi') = \sum_{i=0}^{n} \frac{a_{1i}T_1(\xi')}{(1-\xi'^2)^{1/2}} \quad \sigma_2(\xi'') = \sum_{i=0}^{n} \frac{a_{2i}T_i(\xi'')}{(1-\xi''^2)^{1/2}}. \tag{31}$$

This means that in (29) the first integral may be calculated using the procedure outlined in Section III, while the second, having a continuous kernel, may be computed by using ordinary Gauss–Chebyshev quadrature. The result is $2(n+1)$ equations for the $2(n+1)$ coefficients a_{1i},a_{2i}. It may be verified that for the even mode in which $f_1(x') \equiv 1 \equiv f_2(x'')$ the charge distributions will be related by $\sigma_1(\xi') = \sigma_2(-\xi'')$. The coefficients a_{1i},a_{2i} will be linked by the equations $a_{1i} = (-)^i a_{2i}$ and (29) and (30) will be equivalent. There will thus be just $(n+1)$ equations for the $n+1$ unknown a_{1i}. In the odd mode the corresponding results are

$$\sigma_1(\xi') = -\sigma_2(-\xi') \quad f_1(x') = 1 = -f_2(x'')$$
$$a_{1i} = -(-)^i a_{2i}. \tag{32}$$

Cohn [14] obtained results for the case in which $B/H = 2$. By using conformal mapping he deduced the even and odd impedances

$$Z_0^e = \frac{30\pi}{(\epsilon_{r\,\mathrm{eff}})^{1/2}} \frac{K(k_e')}{K(k_e)} \quad Z_0^o = \frac{30\pi}{(\epsilon_{r\,\mathrm{eff}})^{1/2}} \frac{K(k_o')}{K(k_o)} \tag{33}$$

where

$$k_e = \tanh\left(\frac{\pi}{2}\frac{W}{B}\right) \tanh\left[\frac{\pi}{2}\left(\frac{W+S}{B}\right)\right]$$

$$k_o = \tanh\left(\frac{\pi}{2}\frac{W}{B}\right) \coth\left[\frac{\pi}{2}\left(\frac{W+S}{B}\right)\right]. \tag{34}$$

These results apply to the case in which there are two dielectrics, ϵ_1, ϵ_0, and $\epsilon_1 \neq \epsilon_0$ provided that $\epsilon_{r\,\mathrm{eff}}$ is taken to be $\epsilon_{r\,\mathrm{eff}} = (\epsilon_r + 1)/2$, $\epsilon_r = \epsilon_1/\epsilon_0$.

The impedances are defined as follows: C_0^o and C_0^e are the odd and even capacitances *in vacuo*; C^o,C^e are the odd and even capacitances with dielectric occupying the lower part, as in Fig. 3. Then

$$Z_0^o = \frac{1}{c(C_0^o C^o)^{1/2}} \quad Z_0^e = \frac{1}{c(C_0^e C^e)^{1/2}}$$

where c is the velocity of light *in vacuo*.

Table II shows a comparison between Cohn's results and those obtained from the present analysis for one value of W/B, three of S/B. In each case $n = 4$ was used.

Cohn's conformal mapping results [14] are limited to very particular geometries—equal strips and $B/H = 2$. The present method can immediately be applied to strips of unequal length and $B/H \neq 2$; the only proviso is that the basic Green's function be known. The above examples

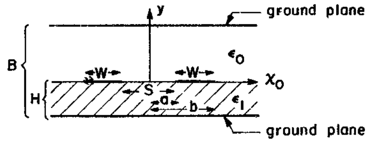

Fig. 3. The geometry of the open couple-strip.

TABLE II
APPROXIMATE VERSUS EXACT IMPEDANCES FOR THE SHIELDED COUPLE-STRIP

W/B	S/B	Z_0^e exact	Z_0^e approx	Z_0^o exact	Z_0^o approx
1	2.0	65.373	65.33	65.319	65.28
1	1.0	65.962	65.92	64.715	64.67
1	0.2	72.155	72.10	55.934	55.88

Note: $B/H = 2$, $\epsilon_r = 1$.

have been test cases to show that the method is versatile and accurate.

Bryant and Weiss [15] considered the coupled stripline in Fig. 4 and computed the capacitances of various configurations. They were aware that their method was inaccurate for small W/H ($\simeq 0.1$) and expected errors less than 1 percent for large W/H ($\simeq 2$). Table III shows a comparison of results obtained from the present method for various values of n. In each case the results have converged by $n = 6$; for $W/H = 0.1$ it is clear that $n = 2$ is sufficient. The table suggests that the expectations of Bryant and Weiss concerning their errors were correct. Bryant and Weiss gave graphs of charge distribution but did not plot dimensionless quantities. Fig. 5 shows the dimensionless quantity $H\sigma(x)/\epsilon_0 V_0$ plotted against dimensionless position. The curves have almost exactly the same shapes as those in [15]. For the particular case $\epsilon_1 = \epsilon_0$, $H \rightarrow \infty$, i.e., for two strips in the open *in vacuo* Sneddon [16] has given an explicit solution for the charge density, namely

$$\sigma(x) = \frac{b}{\pi K(a/b)[(x^2 - a^2)(b^2 - x^2)]^{1/2}}$$

where K is the complete elliptic integral of the first kind. It is found that as b/a increases more terms are needed in the expansion to obtain an accurate charge distribution.

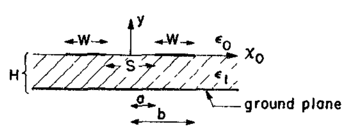

Fig. 4. The charge distribution of an open couple-strip.

TABLE III
THE EVEN AND ODD CAPACITANCES OF THE OPEN COUPLE-STRIP

		Capacitance in [pF/m]				
mode	W/H	N=2	N=3	N=5	N=6	Bryant and Weiss
even	2.0	231.602	232.69	232.615	232.611	231.604
odd	2.0	378.7	358.97	355.639	355.494	351.494
odd	0.1	125.427	125.421	125.420	125.420	120.499
even	0.1	55.510	55.510	55.510	55.510	54.512

Note: $\epsilon_r = 10$, $S/H = 0.2$.

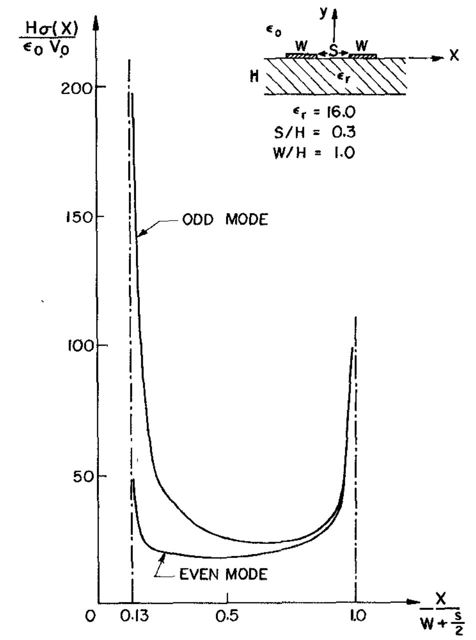

Fig. 5. The geometry of the shielded couple-strip.

Thus for $b/a = 2$, $n = 3$ gives a charge distribution accurate to 3 digits, while $n = 6$ gives 7 digits, $n = 7$ gives 8. For $b/a = 10$, $n = 7$ is required for 3-digit accuracy; for higher n the equations derived from (29) and (30) were found to be ill-conditioned. The case $b/a = 10$ is in any case outside the range of interest.

VI. THE SHIELDED COUPLE-STRIP

As a final application, the method is used to compute the even- and odd-charge distributions and impedances of the shielded couple-strip in Fig. 3 when $\epsilon_1 \neq \epsilon_0$. Fig. 6 shows the impedances as functions of W/H for a number of values of S/H, and $\epsilon_1/\epsilon_0 = 9.6$, $B/H = 4$. It is found that for $S/H = 8$ the even and odd impedances differ only in the fifth decimal place and have the values corresponding to $S/H \rightarrow \infty$. In this limiting case the geom-

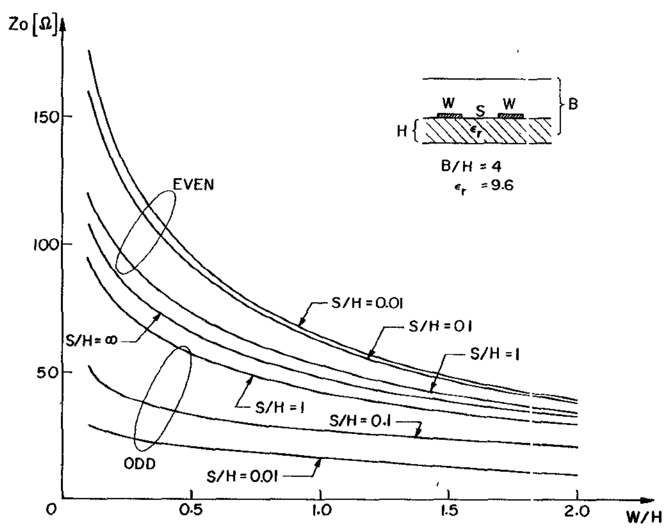

Fig. 6. The variation of the even and odd impedances of the shielded couple-strip with W/H for particular values of S/H.

etry is that of Fig. 2. The case considered by Itoh and Mittra [17] was also computed. Again the charge distribution was found to be identical to theirs.

APPENDIX

With reference to Fig. 2, the covered-microstrip Green's function with the separated logarithmic singularity is from Coen [10]

$$\beta G(x_0,\xi_0) = \ln|x_0 - \xi_0| + \tfrac{1}{2}H(x_0,\xi_0)$$

where

$$\beta = -\pi\epsilon_0(1+\epsilon_r)$$

$$H(x_0,\xi_0) = \ln \frac{4B^2 + (x_0-\xi_0)^2}{\{4H^2 + (x_0-\xi_0)^2\}\{4(B-H)^2 + (x_0-\xi_0)^2\}} + \sum_{n=1}^{\infty} C_n \ln \frac{T_2T_4}{T_1T_3}$$

$$C_n = \frac{n!}{n_1!n_2!n_3!}(-)^{n1}K^{n1+n2}, \qquad n = n_1+n_2+n_3$$

$$K = \frac{1-\epsilon_r}{1+\epsilon_r}$$

$$T = \{2(B-H)(n_1+n_3) + 2H(1+n_2+n_3)\}^2 + \{x_0-\xi_0\}^2$$

$$T_2 = \{2(B-H)(1+n_1+n_3) + 2H(1+n_2+n_3)\}^2 + \{x_0-\xi_0\}^2$$

$$T_3 = \{2(B-H)(1+n_1+n_3) + 2H(n_2+n_3)\}^2 + \{x_0-\xi_0\}^2$$

$$T_4 = \{2(B-H)(n_1+n_3) + 2H(n_2+n_3)\}^2 + \{x_0-\xi_0\}^2.$$

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